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On the Z(4) spin model†

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Abstract. We study the Z(4) spin model in two dimensions in the region of parameters where a critical phase may occur. Our results, obtained by Monte Carlo simulations, Monte Carlo renormalisation group techniques and a study of the elementary excitations near the SOS model, indicate that there is no critical phase in the Z(4) model.

1. Introduction

The Z(4) model is defined, on a square lattice, as follows. The sites are occupied by classical 'spins' S, which can take values $S = e^{i\theta}$, $\theta = 2\pi l/4$, $l = 0, 1, 2, 3$. Assuming nearest-neighbour interactions only, the total energy of the Z(4) model is defined as

$$E = -J_1 \sum_{n,\mu} \{\cos[\theta(n) - \theta(n+\mu)] - 1\} - J_2 \sum_{n,\mu} \{\cos 2[\theta(n) - \theta(n+\mu)] - 1\}. \tag{1}$$

In equation (1), n is a vector that labels the lattice sites, μ represents the conventional primitive vectors of the square lattice, and J_1 and J_2 are the coupling constants.

It is convenient to introduce variables $x_l = \exp(-E_l/kT)$ where E_l is the energy required to rotate a spin by an angle $2\pi l/4$, k is the Boltzmann constant and T is the temperature. With these definitions

$$x_1 = \exp[-(J_1 + 2J_2)/kT] \qquad x_2 = \exp(-2J_1/kT). \tag{2}$$

It can be shown that the partition function of the model obeys generalised self-duality relations in the region $0 < x_1 < (1 + x_2)/2$ and $0 < x_2 < 1$, while it is self-dual along the line

$$2x_1 + x_2 = 1. \tag{3}$$

In the region where generalised self-duality holds, thermodynamical paths can be characterised by the power law $x_2 = x_1^{\alpha}$, $0 < \alpha < \infty$.

The phase diagram of the Z(4) model has been studied by several different formalisms (Wu and Lin 1974, Alcaraz and Köberle 1980, Rujan *et al* 1981, Kohmoto *et al* 1981, Carneiro *et al* 1982, Stavans and Domany 1983, Baltar *et al* 1984). It is found that the Z(4) spin model in two dimensions has at least four distinct phases:

(I) a ferromagnetic phase,
(II) an intermediate partially ordered phase,
(III) a disordered phase,
(IV) a phase with perpendicular order.

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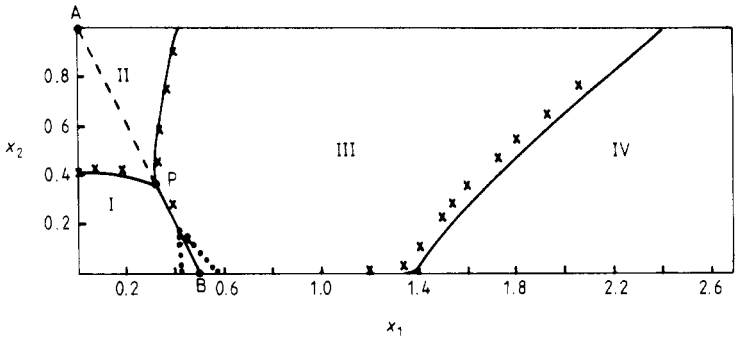

Figure 1. Phase diagram for the Z(4) model. Crosses correspond to Monte Carlo simulations and full lines were obtained by the Migdal-Kadanoff renormalisation group scheme (Baltar *et al* 1984). AB is the self-dual line. Dotted lines represent the boundaries of a hypothetical critical phase, which our calculations indicate does not exist.

These are shown in figure 1, which exhibits results previously obtained by means of Monte Carlo simulations and the Migdal-Kadanoff renormalisation group scheme (Baltar *et al* 1984).

To the phase diagram was added a fifth hypothetical phase, whose possible boundaries are indicated by dotted lines. It corresponds to the conjectured existence of a critical phase, similar to that of $Z(N)$ models with $N \geq 5$, which might appear also in the Z(4) model. This massless phase has been found in the Z(4) gauge model in four dimensions (Creutz and Okawa 1983, Alcaraz and Jacobs 1983). But in spite of the many similarities between $Z(N)$ gauge theories in four dimensions and the corresponding spin theories in two dimensions, it is not clear that their phase diagrams must be identical.

Roberts (1984) has found a critical structure for both Z(4) models, spin and gauge, using perturbative corrections to the Migdal-Kadanoff renormalisation scheme. However, Alcaraz and Drugowich de Felício (1984) studied the spin Z(4) model in two dimensions by means of finite-size scaling techniques and did not find a critical phase.

In this paper we further investigate the question about the existence or not of a critical phase for the Z(4) spin model in two dimensions.

In § 2 we present our results of Monte Carlo simulations in the region where the critical phase may occur, that is, in the region of thermodynamical paths with $\alpha \geq 2$. In § 3 we analyse the elementary excitations of the Z(4) model near the limit of the restricted SOS model.

2. Monte Carlo simulations

We here describe the method of analysis of the Monte Carlo (MC) computer simulations data on square lattices of sizes ranging from 16×16 to 64×64, subjected to periodic boundary conditions.

Both standard MC methods (Binder 1979) and MC renormalisation group (MCRG) techniques (Swendsen 1982) are discussed. The spin-flipping procedure used here is similar to that described in a previous paper (Carneiro *et al* 1982).

In order to obtain a general overview of the phase structure of our system in the region under consideration, we used standard MC methods to study its thermal cycles. For fixed values of the coupling constants, the temperature was varied from a minimum (T_1) to a maximum (T_2) value, and back to T_1 again. At each step the internal energy was measured, and we looked for hysteresis effects, which indicate the existence of a phase transition.

In all thermodynamical paths studied ($\alpha > 2$) a single hysteresis loop was found, indicating a single phase transition. Our results for thermodynamical paths with $\alpha = 5, 10$ and 50, on 20×20 lattices, are shown in figure 2.

We conclude that the analysis of thermal cycles does not indicate the existence of an intermediate massless phase between the ordered and disordered ones.

Using standard MC simulations we calculated the internal energy, specific heat C (from the fluctuations of the internal energy), magnetisation $\langle S \rangle$ and magnetic susceptibility χ (from the fluctuations of $\langle S \rangle$). Peaks in C and χ as functions of temperature, as well as the temperature dependence of $\langle S \rangle$, were used as indications of a phase transition. In these MC simulations, performed on lattices of size 32×32, data were taken after 1 MC step/spin. The total number of MC steps/spin used was 60 000, discarding 20% of the initial steps to allow for thermal equilibrium to be reached.

In figure 3 we show the behaviour of (a) magnetic susceptibility, (b) specific heat and (c) magnetisation as functions of temperature for $\alpha = 10$. The intersection of this thermodynamical path with the self-dual line is at $kT/J_1 = 0.288$. Our results for χ and $\langle S \rangle$ are consistent with a single phase transition localised on the self-dual line.

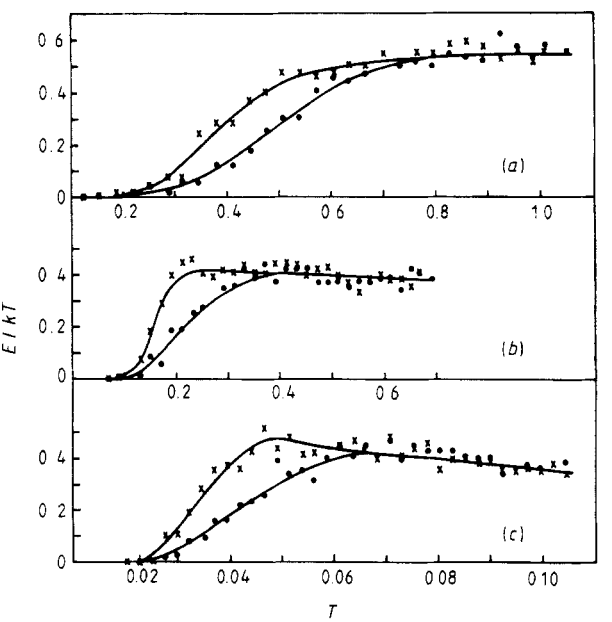

Figure 2. Thermal cycles. Points are plotted after every ten iterations. ●(×) correspond to increase (decrease) in temperature. (a) $\alpha = 5$; the total number of MC steps/spin was 800; T was changed by $\Delta T = 0.0032$ after every iteration. (b) $\alpha = 10$; the total number of MC steps/spin was 650; T was changed by $\Delta T = 0.002$ after every iteration. (c) $\alpha = 50$; the total number of MC steps/spin was 900; T was changed by $\Delta T = 0.000\,24$ after every iteration. Full curves are guides to the eyes.

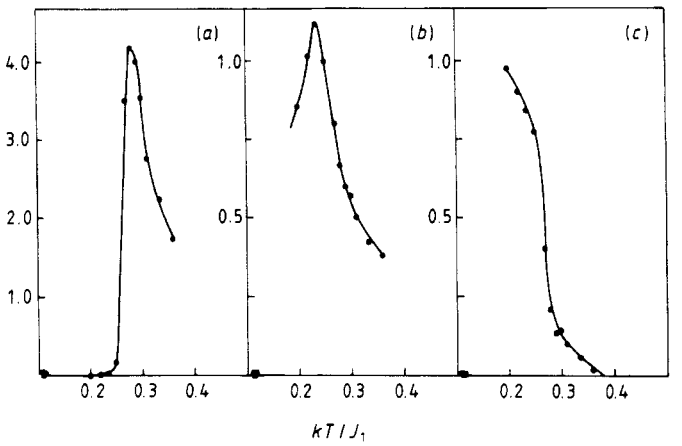

Figure 3. Temperature dependence of (*a*) magnetic suceptibility χ/Nk, (*b*) specific heat C/Nk and (*c*) magnetisation $\langle S\rangle$ for the thermodynamical path $\alpha = 10$.

Although the specific heat shows a peak at a temperature lower than $kT/J_1 = 0.288$, we did not find another peak in the dual region $kT/J_1 > 0.288$, which indicates the existence of a single phase transition.

Similar results were obtained for other thermodynamical paths in the region of interest $\alpha > 2$.

The two-peak structure in the $C \times T$ plot, mentioned in a previous paper (Baltar *et al* 1984) where, using 15 000 MC steps/spin, we obtained a steep and narrow peak close to the self-dual line, and a shorter, broad peak at a higher temperature, was not found in these simulations, involving 60 000 MC steps/spin. The shorter, broad peak is not present, and only the sharp peak, close to the self-dual line, remains.

We also carried out a Monte Carlo renormalisation group analysis of data taken on 16×16, 32×32 and 64×64 lattices. The MCRG method we used consists, as usual (Swendsen 1982), of applying a chosen RG transformation to spin configurations generated by the MC simulation. The RG transformation of scale factor $\lambda = 2$ and the method used to identify phase transitions are described in a previous paper (Baltar *et al* 1985). Correlation functions

$$F_p(r, L) = \sum_{n,\,\mu(p)} \langle \cos[\theta(n) - \theta(n + \mu(p))] \rangle$$

where p denotes first- or second-nearest neighbours, were calculated on lattices of size r^2 ($r = 4, 8$) renormalised from lattices of size L^2($L = 16, 32, 64$). Data were taken after 20 MC steps/spin; again, the total number of MC steps/spin used was 60 000, and 20% of the initial steps were discarded.

In figure 4 we show the behaviour of the second-nearest-neighbour correlation function $F_2(4, L)$ measured on lattices of size 4×4 obtained by renormalisation of lattices of size $L \times L$ ($L = 16$, 32, 64) as functions of temperature for (*a*) $\alpha = 5$, (*b*) $\alpha = 10$ and (*c*) $\alpha = 50$. At low temperatures $F_2(4, 64) > F_2(4, 32) > F_2$ (4,16), which is characteristic of an ordered phase, and at high temperatures $F_2(4, 64) < F_2(4, 32) < F_2(4, 16)$ corresponding to a disordered phase.

The intersections of these thermodynamical paths with the self-dual line are localised at (*a*) $\alpha = 5$; $kT/J_1 = 0.555$, (*b*) $\alpha = 10$: $kT/J_1 = 0.288$ and (*c*) $\alpha = 50$: $kT/J_1 = 0.058$. The results obtained for the correlation functions show that $F_2(4, 64) =$

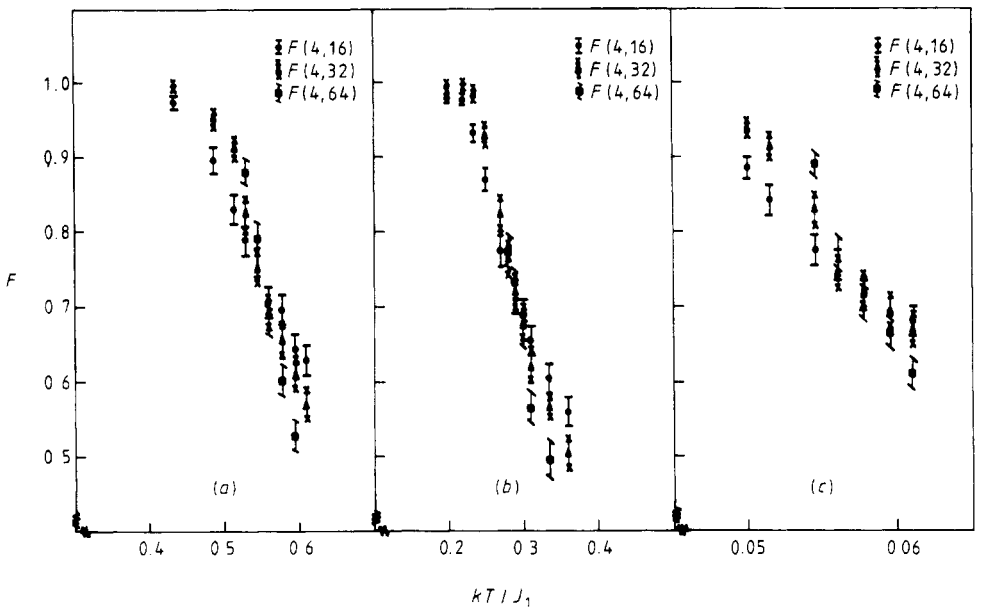

Figure 4. Variation of the correlation functions $F_2(4, 16)$, $F_2(4, 32)$ and $F_2(4, 64)$ with the unrenormalised spin system temperature, for (*a*) $\alpha = 5$, (*b*) $\alpha = 10$ and (*c*) $\alpha = 50$.

$F_2(4, 32) = F_2(4, 16)$ in a very narrow region around the self-dual line, thus indicating a single phase transition.

3. Excitations near the SOS model

On the line $x_2 = 0$ the relative angle $\theta_i - \theta_j$ between two nearest-neighbour spins $\langle ij \rangle$ can take the values $\theta_i - \theta_j = (2\pi/4)(l_i - l_j) = 0, \pm\pi/2$; the value π is forbidden. We can then characterise the lattice configurations for $x_2 = 0$ by the set $\{l_i\}$, such that $|l_i - l_j| = 0, 1$. In this case, the Z(4) model becomes equivalent to the restricted SOS model (Domany *et al* 1980, den Nijs 1985).

Another way of characterising the configurations is to draw vertices in the centre of elementary plaquettes with arrows between nearest-neighbour spins pointing outwards (inwards) for $l_i - l_j = 1$ (-1) or zeros for $l_i - l_j = 0$.

For $x_2 = 0$ the allowed vertex configurations are those corresponding to the eight-vertex model ($|l_i - l_j| = 1$ for all spins of an elementary plaquette) with vorticity 0, ± 4, and those with four zeros ($l_i - l_j = 0$ for all spins of an elementary plaquette) or two zeros and two single arrows with vorticity zero.

For $x_2 \neq 0$ ($x_2 \ll 1$), configurations with $|l_i - l_j| = 2$ are also allowed ($|\theta_i - \theta_j| = \pi$) and they can be characterised in the vertex picture by double arrows. Vertices containing one double arrow can also have vorticity 0, ± 4. The combination of two such vertices defines a vortex of vorticity ± 4, as shown in figures 5(*a*) and (*b*).

In the ferromagnetic phase near the restricted SOS model ($x_2 \ll 1$), the most favoured configuration of the lattice is a background sea of zeros ($l_i - l_j = 0$) with excitations such as closed loops of single arrows, impurities and vortex-antivortex pairs linked together linearly by strings of zeros (figure 5(*c*)) (den Nijs 1985). In the disordered

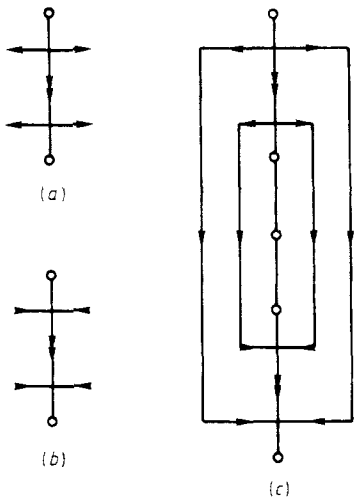

Figure 5. Vertex diagrams of (*a*) vortex of vorticity +4, (*b*) antivortex of vorticity −4, (*c*) pair of vortex-antivortex.

phase, vortices are free. It may happen that the ferromagnetic ordered phase melts via an intermediate critical phase, by means of two phase transitions.

If a transition to a critical phase is present in the Z(4) model near the $x_2 = 0$ line, then the lattice configurations in this phase should be similar to those of the ferromagnetic phase, but with logarithmic (instead of linear) interaction between vortices and antivortices.

On the $x_2 = 0$ line, this should correspond to the roughening transition of the restricted SOS model. Above the roughening temperature T_R, the rough interface can be characterised by its effective Gaussian coupling constant K (Kadanoff and Brown 1979, Knops 1980, Pruisken and Brown 1981), which has the universal value $K_R = \pi/2$ at T_R (Kosterlitz 1974, Knops 1977). At the self-dual line, $K^* = 2\pi/N$ (José *et al* 1977). Increasing the temperature further, a Kosterlitz-Thouless transition into a disordered phase takes place at $K_{KT} = 8\pi/N^2$ (Kosterlitz and Thouless 1973, José *et al* 1977).

Near the $x_2 = 0$ line, for the Z(4) model, this should correspond to a situation in which the positional entropy of the composite vortices exceeds the logarithmic interaction, leading to a disordered phase with free vortices and antivortices (Kosterlitz-Thouless transition).

However, we observe that for the Z(4) model on the $x_2 = 0$ line $K_{KT} = K_R = K^* = \pi/2$, indicating the absence of an intermediate phase between the ordered and disordered ones—a single phase transition occurs, on the self-dual line.

We then conclude that the study of the excitations near the restricted SOS model strongly supports the evidence that there is no critical phase in the Z(4) spin model in two dimensions and enables us to locate the roughening transition of the restricted SOS model.

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